Pressure is force per unit area, $d\vec{F} = -pd\vec{A}$, where the direction of $\vec{A}$ is outward along the surface normal.

The compressibility of a fluid $k = -(1/V)\partial V/\partial p$. $k = 1/p$ for an ideal gas, but is much smaller for ordinary liquids, which can be regarded as approximately incompressible ($k \approx 0$).

Pressure variation in a column of incompressible liquid $p(z) = p_0 + \rho g(h - z)$ where h is the height of the column. The gradient of the pressure gives the *force per unit volume* $\vec{f} = -\nabla p$. The volume integral of $\int_{\mathcal{V}} \vec{f} \, dV$ is the hydrostatic force on the material in the enclosed volume.

Pascal's principle: An external pressure is transmitted undiminished to all parts of a fluid in equilibrium. Note that this is a statement about the pressure, not the force.

Archimedes principle : The bouyant force on an object in a fluid is the weight of the excluded fluid.

Force on curved surface: The force from a fluid with (constant) pressure p on a hemisphere of radius R is $\vec{F} = \oint_{\mathcal{S}} p d\vec{A} = \pi p R^2$. In general, if the pressure is constant, this force depends only on the cross sectional area perpendicular to the line of action of the force.

Divergence Theorem: For any vector field $\vec{u}(\vec{r})$, the flux of $\vec{u}$ through a closed surface $\mathcal{S}$ is $\oint_{\mathcal{S}} \vec{u} \cdot d\vec{A} = \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{u} \, dV$. This means the total flow of $\vec{u}$ through the closed surface is the sum of the net flows from each point (aka the *divergence* of $\vec{u}$) contained in the volume bounded by the surface. (Worked example: a calculation of the Archimedes force from the pressure of a fluid at the boundary of an immersed object).

Surface Tension: The surface energy of a fluid is $U_s = \gamma A$ where A is the surface area and γ is the surface tension. $[\gamma] = \text{J/m}^2 = \text{N/m}$. The latter is useful for formulating the surface tension force using the contact "line".

Equation of Continuity: For a steady flow bounded by a container of varying cross sectional area A, the volume flow rate $\Phi = vA$ is constant.

Equation of Motion: The force density $\vec{f} = \rho \partial \vec{v}/\partial t + \rho(\vec{v} \cdot \vec{\nabla})\vec{v}$, where the second term is called the "convective derivative." For steady flow we have $\vec{f} = \rho(\vec{v} \cdot \vec{\nabla})\vec{v}$. In these expressions $\vec{v}$ is the average velocity of all the particles found in a fixed volume element of the fluid.

Note: for steady flow in a circular channel $(\vec{v} \cdot \vec{\nabla})\vec{v} = -(v^2/r)\hat{r}$ in each volume element. This gives the net "centripetal acceleration" of the particles in the volume element. Since the flow is steady $\partial \vec{v}/\partial t = 0$.

Bernoulli's Law: For steady, incompressible and irrotational flow, $(1/2)\rho v^2 + \rho g z + p = $ constant. This is the work energy theorem for steady flow in a fluid.

Torricelli's Law: The flow velocity from a liquid column of height h is the free fall velocity $v = \sqrt{2gh}$.

The viscous force transmitted across a surface $\mathcal{A}$ of a flowing liquid is $F/\mathcal{A} = \eta\dot{\varepsilon}$ where $\dot{\varepsilon}$ is the strain rate and η is the viscosity.

Poiseuille's Law: The flow velocity of a viscous fluid in a circular pipe with radius R and length ℓ is *nonuniform*

$$v(r) = \frac{\Delta p}{4\eta\ell}\left(R^2 - r^2)\right) \tag{1}$$

and the volume flow rate is proportional to R^4

$$\Phi = \frac{\pi R^4 \Delta p}{8\eta\ell} \tag{2}$$

Stokes' Law: A sphere of radius R moving at velocity v relative to a fluid with viscosity η experiences a viscous drag force $F_v = 6\pi\eta R v$.

The pressure from a gas on the walls of its container is calculated from kinetic theory $p = 2n\langle K\rangle/3$ where n is the number of particles per unit volume and $\langle K\rangle$ is the mean kinetic (translational) kinetic energy. This gives a microscopic basis for the ideal gas equation of state $p\mathcal{V} = NkT = N_m \mathcal{R} T$ where k is Boltzmann's constant, and $\mathcal{R}$ is the gas constant.

Two Equilibria: Mechanical equilibrium between two otherwise isolated systems that are free to exchange volumes is established when the pressures are equal. Thermal equilibrium between two systems that can exchange energy is established when the temperatures are equal.

Heat I: When two otherwise isolated systems at different temperatures are in thermal contact, heat flows from the high temperature to low temperature system. This is a nonequilibrium state. Warning: do not confuse temperature and heat.

Heat II: The heat capacity of an object is the ratio of the heat required to raise the temperature by ΔT to the temperature change, $C = Q/\Delta T$ in the limit that $\Delta T \to 0$. The heat capacity is an extensive quantity that is proportional to the total number of particles in the system. It is useful to re-express this as a specific heat $c = C/N$ or a molar heat capacity $c^m = C/N_m$. For compressible states (like a gas) we need to specify the thermodynamic state of the system while the heat is added, by distinguishing between the specific heats at constant volume and at constant pressure.

Specific Heats for Ideal Gases For a monatomic ideal gas, only the translational motions contribute to the thermal energy so that the molar heat capacity $c_v^m = 3R/2$ where R is the universal gas constant. By the equipartition theorem each "thermally accessible" degree of freedom contributes $kT/2$ to the internal energy so that, for example, an ideal diatomic gas has a molar heat capacity $c_v^m = 5R/2$. In general the heat capacity can depend on the temperature since at high temperature more internal motions become thermally accessible.

Heat III: The rate of flow of heat $\dot{Q}$ through a rectangular parallelpiped of cross sectional area $\mathcal{A}$ and width d is

$$\dot{Q} = \frac{k\mathcal{A}\Delta T}{d} \tag{3}$$

where k is the thermal conductivity, and ΔT is the temperature difference across the width of the parallelpiped.

Three Maxwell Distributions: For a one dimensional ideal gas, the velocity distribution $f_1(v_x)$ is

$$f_1(v_x) = n_1 \sqrt{\frac{m}{2\pi kT}} \exp(-\frac{mv_x^2}{2kT}) \tag{4}$$

where n_1 gives the one dimensional density (number of particles per unit length). $f_1(v_x)\Delta v_x$ gives the number of particles per unit length with velocities between v_x and $v_x + \Delta v_x$. For a three dimensional gas this generalizes to

$$f(v_x, v_y, v_z) = n \left(\frac{m}{2\pi kT}\right)^{3/2} \exp(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}) \tag{5}$$

where n is the three dimensional density (number of particles per unit volume). The distribution of molecular *speeds* $g(v)$ can be derived from this, giving

$$g(v) = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \tag{6}$$

$g(v)\Delta v$ gives the number of particles per unit volume with *speeds* between v and $v + \Delta v$. The most probable speed occurs when $dg/dv = 0$, so $v_p = \sqrt{2kT/m}$. The mean square speed $\langle v^2 \rangle$ is obtained by an average over this distribution

$$\langle v^2 \rangle = \frac{\int_0^\infty dv\, v^2\, g(v)}{\int_0^\infty dv\, g(v)} = \frac{3kT}{m} \tag{7}$$

(Note: the proof of the last equality is assigned as a homework problem.)

Boltzmann Factor: In thermal equilibrium at temperature T the probability to find a molecule in a state with energy E (note this is the sum of its kinetic and potential energies) is proportional to the "Boltzmann factor" $\exp(-E/kT)$. This means that in equilibrium very high energy states $E >> kT$ have exponentially small probabilities.

Useful Gaussian Integrals:

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx = \sqrt{\pi/\alpha}$$

$$\int_0^\infty \exp(-\alpha x^2) x\, dx = 1/2\alpha$$

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2) x^{2n}\, dx = (-1)^n \frac{d^n}{d\alpha^n} \sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty \exp(-\alpha x^2) x^{2n+1}\, dx = (-1)^n \frac{d^n}{d\alpha^n} \frac{1}{2\alpha} \tag{8}$$

First Law of Thermodynamics: $\Delta U = Q + W$ where U is the internal energy, Q is the heat flow to the system ($Q < 0$ is heat flow out and $Q > 0$ is heat flow in) and W is the work done *on the system.* For a compressible system $dW = -pdV$.

More on Heat Capacities: $Q = C\Delta T$, but for a compressible system, we need to distinguish the thermodynamic state of the system while the heat is added. Thus C_V is the heat capacity at constant *volume* ($= 3Nk/2$ for an ideal monatomic gas) and C_p is the heat capacity at constant *pressure* ($= 5Nk/2$) for an ideal monatomic gas. Any quasistatic change of state can be reconstructed from combinations of infinitesimal constant volume and constant pressure changes of state. It is useful to define the dimensionless thermodynamic parameter $\gamma = C_p/C_V$.

State Functions A state function depends only on the equilbrium thermodynamic state of a system, not its history. An example is the internal energy, which for an ideal gas depends only on the temperature and the total number of molecules, i.e. $U(T) = 3NkT/2$ for an ideal monatomic gas of N atoms. The internal energy cannot change in any cyclic process that returns the system to its initial equilbrium state.

Path Functions Heat added to/extracted from and work done to/by a thermodynamic system in a quasistatic change of state can depend on the "path" that connects the initial and final states. In particular the total work done on, and heat transferred to a thermodynamic system in a quasistatic cycle can be nonzero.

Some Quasistatic Changes of State

Isothermal Compression/Expansion (Ideal Gas): $p\mathcal{V} = $ constant, $\Delta U = 0$, $W = -NkT \log(\mathcal{V}_f/\mathcal{V}_i)$, $Q = NkT \log(\mathcal{V}_f/\mathcal{V}_i)$.

Isobaric Compression/Expansion (Ideal Gas): $\Delta U = Q + W$, $W = -p\Delta \mathcal{V}$, $Q = C_p \Delta T$.

Adiabatic Compression/Expansion (Ideal Gas): $p\mathcal{V}^\gamma = $ constant (or $T\mathcal{V}^{\gamma-1} = $ constant$'$), $Q = 0$, $\Delta U = W = (p_f \mathcal{V}_f - p_i \mathcal{V}_i)/(\gamma - 1)$. These formulas hold for a reversibly adiabatic change of state (we refer to this change as *fast but not too fast*).

Entropy: is a quantitative measure of the disorder of an equilibrium thermodynamic state. It is an extensive quantity, proportional to the number of atoms/molecules in a system. *Disorder* in this context is a logarithmic measure of the number of states that store the total thermal energy U in equilibrium at temperature T. Entropy is a state function, i.e. an equilibrium thermodynamic state has a definite entropy. For example, the entropy of the ideal gas at temperature T, pressure p and volume $\mathcal{V}$ is

$$S = Nk \log(U^{3/2}\mathcal{V}) + \text{constant} \tag{9}$$

where the constant is called the chemical constant (it is independent of T and $\mathcal{V}$). Other formulas can be derived for other thermodynamic systems. The change of entropy during a thermodynamic change of state depend only on the initial and final states, and not on the history during the change of state. The change of entropy can be evaluated by direct calculation from equation (9) (we call this the *state function method*) or by evaluating the integral *the heat formula*

$$\Delta S(a \to b) = \left[\int_a^b \frac{dQ}{T} \right]_{quasistatic} \tag{10}$$

on any quasistatic (reversible) path that connects the states a and b. This latter method is the *classical* definition of the change in entropy (it predates the statistical interpretation.)

Second Law of Thermodynamics: Only processes with $\Delta S_{tot} \geq 0$ occur in thermodynamic systems. To properly use this formula *be sure to include the changes of entropy of the reservoirs with which the system interacts!*. Note that the entropy of a single part of an interacting system (sometimes called a subsystem) can decrease in a thermodynamic change of state (an example is the entropy change of a gas in an isothermal compression). Reversible processes obey the equality $\Delta S_{tot} = 0$. In practice there are no truly reversible processes, since every quasistatic change involves many infinitesimal departures from and relaxations back to equilibrium states (and every such relaxation is a one way process where $\Delta S > 0$).

Two adiabatic expansions. A reversibly adiabatic expansion has $Q = 0$, and using the heat formula has $\Delta S = 0$. This is called an isentropic change. An adiabatic free expansion (for example breaking the partition between a confined gas and a vacuum) involves the relaxation from an initial highly nonequilibrium state to a final equilibrium state. This is an irreversible process that has $\Delta S \neq 0$ despite the fact that $Q = 0$. Notice that the heat formula cannot be applied (directly) to this process since the system evolves through a sequence of nonequilibrium intermediate states to which equation (10) is inapplicable.

Short course in Thermodynamics

(a) Understand the three step cycle: isothermal (slow) compression, reversible adiabatic (fast but not too fast) expansion, and reheating at constant volume to return to initial state.

(b) Understand the difference between the reversible adiabatic expansion and the adiabatic free expansion (and be able to calculate the change of entropy for each.)

(c) Understand that a thermodynamic cycle can *partially* convert *heat* into *work*. No cycle can completely convert heat into work. This is the *Kelvin statement* of the second law of thermodynamics.

Efficiency of Thermodynamic Cycles The efficiency of a cyclic thermodynamic process is $\eta = W/Q_{in}$ where W is the net work done and Q_{in} is the heat absorbed. The optimum efficiency is realized by a theoretical cycle called the *Carnot cycle* that extracts heat from a high temperature reservoir at temperature T_{high} does work and discharges heat to a low temperature reservoir T_{low}. The Carnot cycle is a reversible cycle that absorbs heat only at its highest operating temperature and discharges heat only at its lowest operating temperature. The Carnot cycle has an efficiency $\eta = W/Q_{in} = (Q_{in} - Q_{out})/Q_{in} = 1 - T_{low}/T_{high}$.

Harmonic Oscillator Review The equation of motion for the free harmonic oscillator with viscous damping is

$$m\ddot{s} + b\dot{s} + ks = 0 \tag{11}$$

The solutions are most naturally represented by superpositions of the (in general) complex solutions $\tilde{s} \propto \exp(pt)$ where

$$p_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{12}$$

For example, the motion of the undamped oscillator is represented

$$s(t) = \tilde{a}e^{i\omega_o t} + \tilde{a}^* e^{-i\omega_o t} = 2|a| \cos(\omega_o t - \delta) \tag{13}$$

where $\delta = \arctan(\dot{s}(0)/\omega s(0))$ and $2|a| = \sqrt{s(0)^2 + \dot{s}(0)^2/\omega_o^2}$.

The solutions to the EOM are damped complex exponentials (these describe damped oscillations) when $b/2m < \sqrt{k/m}$ but are damped real exponentials (and thus not oscillating) when $b/2m > \sqrt{k/m}$. In the strongly overdamped limit the slower decaying exponential has a rate constant $\approx k/b$ that does not depend on the mass. When $b = 2\sqrt{mk}$ the two roots of the characteristic equation $p_{\pm}$ are equal, and the solutions are called *critically damped*. The critically damped solutions can be expressed as combinations of two linearly independent solutions

$$s(t) = (A + Bt)e^{-bt/2m} \quad \text{critically damped} \tag{14}$$

Driven Oscillations The equation of motion

$$M\ddot{s}(t) + b\dot{s}(t) + ks(t) = kW(t) = kW_o \cos(\omega t) \tag{15}$$

describes a damped oscillator whose motion is driven externally at frequency ω, in this case by harmonically displacing a wall to which a Hooke's law spring is attached. After a transient period (the duration of the transients are determined by the damping constant) the block oscillates at the same frequency as the drive, but shifted in phase

$$s(t) = S_o \cos(\omega t - \beta) \tag{16}$$

where $S_o = \omega_o^2 W_o/\sqrt{(\omega_o^2 - \omega^2)^2 + \omega^2 \gamma^2}$ and $\beta = \arctan(\omega\gamma/(\omega_o^2 - \omega^2))$. When driven at very low frequency $\omega << \omega_o$ the oscillator moves in phase with the driving force and at nearly the same amplitude. When driven at very high frequency $\omega >> \omega_o$ the oscillator moves out of phase with the driving force, with a small response limited by its inertial mass. The largest response occurs at the resonant frequency $\omega = \omega_o$ where $S_o/W_o = \omega_o/\gamma \equiv Q$, so that Q gives the *amplification factor* on resonance. The Q factor *also* characterizes the rate of energy decay of a free oscillator: it is the number of oscillations (measured in radians) for the mechanical energy in the free oscillator to decay to $1/e$ of its initial value. The Q factor *also* characterizes the frequency

response of the driven oscillator: it gives the dimensionless full width $Q = \omega_o/\Delta\omega$ measured on the response curve at the points where the steady state amplitude is reduced from its maximum value (on resonance) by a factor of $1/\sqrt{2}$. A typical ball and string pendulum has $Q \approx 10$ (although often used for discussion, it is not a particularly good oscillator), while a tuning fork has $Q \approx 10^4$.

Coupled Oscillators The coupled motions of the masses in the mass spring network shown below can be represented as an expansion in the normal modes of the system.
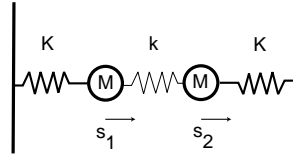
(1) motion with equal displacements of the two masses (equal amplitudes and in phase) at a frequency $\omega_1 = \sqrt{k/m}$.

(2) motion with antisymmetric displacments of the two masses (equal amplitudes, but out of phase) at a frequency $\omega_2 = \sqrt{3k/m}$.

Using these fundamental modes the motions of the masses can be reconstructed

$$
\begin{aligned}
s_1(t) &= \frac{1}{2}\left(A\cos(\omega_1 t + \alpha) + B\cos(\omega_2 t + \beta)\right) \\
s_2(t) &= \frac{1}{2}\left(A\cos(\omega_1 t + \alpha) - B\cos(\omega_2 t + \beta)\right)
\end{aligned}
\tag{17}
$$

where the constants A, B, α and β are determined by the initial conditions of the motion. When the system is set into motion with an initial displacement of one of the masses (say mass 1 with initial displacement C) the solutions are

$$
\begin{aligned}
x_1(t) &= C\cos(\bar{\omega}t)\cos(\Delta\omega t/2) \\
x_2(t) &= C\sin(\bar{\omega}t)\sin(\Delta\omega t/2)
\end{aligned}
\tag{18}
$$

where $\bar{\omega} = (\omega_1 + \omega_2)/2$ and $\Delta\omega = \omega_2 - \omega_1$. This determines the frequency of the "fast" oscillation, and the "slow" modulation of the amplitude if $\omega_2 \approx \omega_1$. Since the mechanical energy $\propto$ the square of the oscillator amplitude, the energy oscillates at twice the frequency of the displacements.

Coupled Oscillators as an Eigenvalue Problem The normal modes can be determined as the solutions of an eigenvalue problem. Introducing a column vector X that gives a fixed combination of displacements that evolve in time with a common time dependence

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \\ . \end{pmatrix} e^{pt} \tag{19}$$

the equation of motion in matrix form is

$$\ddot{X}(t) = p^2 \begin{pmatrix} x_1 \\ x_2 \\ . \end{pmatrix} e^{pt} = \begin{pmatrix} d_{11}/M_1 & d_{12}/M_1 & ... \\ d_{21}/M_2 & d_{22}/M_2 & ... \\ . & . & ... \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \end{pmatrix} e^{pt} \tag{20}$$

where the d_{ij} are determined by the network of springs that couple the masses. This EOM has nonzero solutions only if the *determinant*

$$\mathrm{Det} \begin{pmatrix} -p^2 + d_{11}/M_1 & d_{12}/M_1 & ... \\ d_{21}/M_2 & -p^2 + d_{22}/M_2 & ... \\ . & . & . \end{pmatrix} = 0 \tag{21}$$

The roots of this polynomial (it is of order $2N$ for a problem with N coupled masses) give the (complex) rate constants p, and once these are known equation (19) is solved for the amplitude ratios (e.g. x_2/x_1, etc.) for each mode of the system. The observed motions of the masses are then given by a linear superposition of these fundamental motions, with coefficients specified by the initial conditions of the motion. Note that there are $2N$ initial conditions required to specify the solution to the N particle problem. (For example the two mass problem requires four constants, that could be determined knowing the initial displacements and velocities of each of the masses.)

Continuum Limit for N Coupled Masses For N-coupled masses each of mass M with average separation a and with each connected to its nearest neighbors by Hooke's Law springs with spring constants K, the equations of motion are second order difference equations

$$M\ddot{s}_n = K s_{n-1} - 2K s_n + K s_{n+1} \tag{22}$$

Solutions that are slowly varying with the lattice index n can be obtained by passing to the continuum limit $s_n \rightarrow s(na) \equiv s(x)$ and satisfy the one dimensional wave equation

$$M\ddot{s} = Ka^2 \frac{\partial^2 s}{\partial x^2} \tag{23}$$

Similar EOM's describe the motion of small transverse oscillations of a stretched string, and the small density fluctuations in a gas contained in a narrow pipe, and the transverse fluctuations of the electric and magnetic fields that are constrained to vary only along a single direction in space, etc. For example the wave equation for transverse displacements $h(x,t)$ on a stretched string is

$$\ddot{h}(x,t) = \frac{T}{\mu} \frac{\partial^2 h(x,t)}{\partial x^2} \tag{24}$$

where T is the string tension and μ is the mass per unit length.

Solution by Propagation (D'Alembert's method). In one dimension the solutions to the wave equation are superpositions of the forward and backward propagating solutions

$$h(x,t) = f(x - vt) + g(x + vt) \tag{25}$$

where the first term on the right hand side gives a waveform of shape f that propagates with no change of shape to the right, and the second term gives a waveform of shape g that propagates with no change of shape to the left. The two components can be isolated by studying the initial conditions specified by $h'(x,0) = \partial h(x,0)/\partial x$ and $\dot{h}(x,0) = \partial h(x,t=0)/\partial t$. This gives the composition rule

$$
\begin{aligned}
f'(x) &= \frac{1}{2}\left(h'(x,0) - \frac{1}{v}\dot{h}(x,0)\right) \\
g'(x) &= \frac{1}{2}\left(h'(x,0) + \frac{1}{v}\dot{h}(x,0)\right)
\end{aligned}
\tag{26}
$$

Equations (26) can be integrated to determine the two components f and g, and this provides a complete solution to the one dimensional equation of motion. This method is extremely powerful for problems in one dimensional wave motion, but not generalize naturally to wave propagation in higher dimensions.

Solution by Separation of Variables The wave equation can also be solved by factoring the space and time dependence of it's solutions $h(x,t) = A(x)B(t)$. This leads to two decoupled equations for the space and time factors that are linked by the "separation constant" C

$$
\begin{aligned}
\frac{\partial^2 A(x)}{\partial x^2} &= \frac{\mu}{T}CA(x) = -\frac{\omega^2}{c^2}A(x) = -k^2 A(x) \\
\frac{\partial^2 B(t)}{\partial t^2} &= CB(t) = -\omega^2 B(t)
\end{aligned}
\tag{27}
$$

The solutions are the periodic functions $A(x) = A_o\cos(kx+\alpha)$ and $B(t) = B_o\cos(\omega t+\beta)$. Thus the factored solution of the wave equation is

$$h(x,t) = H_o\cos(kx + \alpha)\cos(\omega t + \beta) \tag{28}$$

Not all solutions of the wave equation can be expressed in separable (factored) form, but sums of factored solutions can be used to construct a general solution to the wave equation. This is the subject of the first homework exercise on problem set 10.

Solution to the Wave Equation in a Fourier Series For a one dimensional string of length L, fixed at both its endpoints, the boundary conditions for solutions of the wave equation are $h(0,t) = h(L,t) = 0$. Solutions to the wave equation can be written in a Fourier series

$$h(x,t) = \sum_m a_m\sin(k_m x)\cos(\omega_m t + \beta_m) \tag{29}$$

where $k_m = m\pi/L$ and $\omega_m = ck_m = m\pi c/L$. Thus the k's for the allowed "modes" of the string are *quantized*. The coefficients a_m and phase shifts β_m are determined by the initial conditions $h(x,0)$ and $\dot{h}(x,0)$ specifying the initial state of the string motion. Thus for a string released from rest $\dot{h}(x,0) = 0$ so that $\beta_m = 0$ and the a_m can be extracted from an integral over the length of the string (i.e. using *Fourier's trick*).

$$a_n = \frac{2}{L} \int_0^L h(x,0) \sin(\frac{n\pi x}{L})\, dx \tag{30}$$

The expansion coefficients are then substituted into Eqn (29) to describe the subsequent motion of the string. Analogous series can be developed for other initial conditions.

Energy Densities The distribution of kinetic and potential energies in a wave are nonuniform in space and time, and can be described by the *energy densities* (this is an energy per unit length in one dimension)

$$
\begin{aligned}
\mathcal{K}(x,t) &= \frac{\mu}{2}\left(\frac{\partial h}{\partial t}\right)^2 \\
\mathcal{U}(x,t) &= \frac{T}{2}\left(\frac{\partial h}{\partial x}\right)^2 \\
\text{and } E &= \int (\mathcal{K} + \mathcal{U})\, dx
\end{aligned}
\tag{31}
$$

For the *special case* of a pure propagating wave of general shape, i.e. with $h(x,t) = f(x - ct)$ or $h(x,t) = g(x + ct)$ (note that this *excludes* the superposition solution $h(x,t) = f(x-ct)+g(x+ct)$), the energy densities $\mathcal{K}(x,t) \equiv \mathcal{U}(x,t)$ at each space-time point (x,t).

Some Useful Results for Gaussian Pulses

For a Gaussian pulse shape $h(x) = h_o \exp(-x^2/\sigma^2)$, the potential energy density is

$$\mathcal{U}(x) = \frac{2Th_o^2}{\sigma^4}x^2 e^{-2x^2/\sigma^2} \tag{32}$$

and the total stored potential energy is

$$U = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{Th_o^2}{\sigma} \tag{33}$$

Energy Transport in a One Dimensional Harmonic Wave For a right moving harmonic wave $h(x,t) = h_o \cos(kx - \omega t)$ the transmission of energy past a point x is given by the *wave power*

$$\mathcal{P}(x,t) = c\mathcal{E}_1 = \mu c\omega^2 h_o^2 \sin^2(kx - \omega t) \tag{34}$$

where $\mathcal{E}_1 = \mathcal{K} + \mathcal{U}$ is the energy density (with dimensions energy/length in one dimensional wave motion). The average power is half its maximum value $\bar{\mathcal{P}} = \mathcal{P}_{\max}/2$.

In three dimensions it is useful to introduce the wave *intensity* which is the *power per unit area* , $c\mathcal{E}_3$ where $\mathcal{E}_3$ is the three dimensional energy density (an energy per unit volume). Thus for three dimensional wave propagation the total power transmitted through a closed surface is the flux of the wave intensity. The Poynting vector of electrodynamics gives the intensity of the electromagnetic wave.

Reflection and Transmission Coefficients For a wave incident from a medium with propagation speed c_1 onto a medium with propagation c_2, the incident wave is scattered into a reflected and transmitted wave. The transmitted and reflected amplitudes are related by the transmission and reflection coefficients

$$
\begin{aligned}
r &= \frac{h_r}{h_o} = \frac{c_2 - c_1}{c_2 + c_1} \\
t &= \frac{h_t}{h_o} = \frac{2c_2}{c_2 + c_1}
\end{aligned}
\tag{35}
$$

Note that $-1 < r < 1$ while $0 < t < 2$. A negative value of r indicates that the amplitude is inverted when the pulse is reflected. The situation with $t > 1$ is discussed in last assignment on Homework Set 11.

Conservation of energy implies that the incident power equals the sum of the reflected and transmitted wave powers, so that

$$
\mathcal{P}_i = \mathcal{P}_r + \mathcal{P}_t
\tag{36}
$$

For a Gaussian pulse (or for any finite wavepacket), integrating this expression with respect to time shows that the total energy is conserved, i.e.

$$
E_i = E_r + E_t
\tag{37}
$$

For a harmonic wave (that is infinite in extent), the power expression can be integrated over one period of oscillation to demonstrate that the incident energy equals the sum of the reflected and transmitted energies.

Superposition and Propagation of Wave Packets

The superposition of harmonic waves with frequencies in the range $\bar{\omega} - \Delta\omega/2 < \omega < \bar{\omega} + \Delta\omega/2$ produces a *wave packet*, a propagating disturbance that is localized in space and time. For example, a solution to the wave equation

$$
h(x, t) = \sum_{n=1}^{N} h_n \cos(k_n x - \omega_n t)
\tag{38}
$$

where $h_n = h_o/N$ in the allowed range of frequencies. This sum can be converted to an integral

$$
h(x, t) = \frac{h_o}{\Delta\omega} \int_{\bar{\omega} - \Delta\omega/2}^{\bar{\omega} + \Delta\omega/2} d\omega \ \cos(k(\omega)x - \omega t)
\tag{39}
$$

Since $k(\omega) \approx \bar{k} + (\partial k/\partial\omega)(\omega - \bar{\omega})$, the integral over (a narrow range of) ω gives

$$
h(x, t) = h_o \cos(\bar{k}x - \bar{\omega}t) \times \frac{\sin\left(\frac{\partial k}{\partial\omega}\frac{\Delta\omega}{2}\left(x - \frac{\partial\omega}{\partial k}t\right)\right)}{\frac{\partial k}{\partial\omega}\frac{\Delta\omega}{2}\left(x - \frac{\partial\omega}{\partial k}t\right)}
\tag{40}
$$

This is the product of two forward propagating solutions: the cosine factor propagates in the positive direction at a velocity $c_p = \bar{\omega}/\bar{k}$. c_p is the phase velocity and gives the speed of *crests* in the waveform. The second $\sin(\xi)/\xi$ factor propagates in the positive direction at a velocity $c_g = \partial\omega/\partial k$. This is the group velocity, and gives the speed of the *entire packet*. The packet has a temporal width $\Delta t = 2\pi/\Delta\omega$, so that the narrower the spread of frequencies the wider is the resulting the wave packet. For a nondispersive medium (like the ideal stretched string) $c_p = c_g$ and the crests move at the same velocity as the packet. However, dispersive media exist (e.g. light propagating in matter, sound waves propagating in a solid, water waves propagating at a surface, etc.), so that in general the propagation is characterized by two speeds, one for the crests and one for the packet. Of these two the group velocity c_g is more useful, since it describes the rate of propagation of the energy. In engineering the relation $\Delta\omega\Delta t = 2\pi$ is sometimes called the *bandwidth theorem*, and in quantum mechanics it is the *uncertainty principle*. It implies that a pulse of finite duration cannot have a sharp frequency and conversely that a superposition of broad band of frequencies is required to produce a temporally sharp pulse.